An analogue of the strengthened Hanna Neumann conjecture for virtually free groups

Alexander Zakharov
University of Wroclaw

Joint work with Anton Klyachko

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Theorem (Hanna Neumann, 1957)

For any subgroups A and B of a free group F

$$\overline{rank}(A \cap B) \leq 2 \cdot \overline{rank}(A) \cdot \overline{rank}(B),$$

where $\overline{rank}(H) = max(0, rank(H) - 1)$ *is the reduced rank of a free group H.*

Theorem (Friedman, Mineyev, 2011; known before as Hanna Neumann Conjecture)

For any subgroups A and B of a free group F

$$\overline{rank}(A \cap B) \leq \overline{rank}(A) \cdot \overline{rank}(B).$$

Moreover, for any system of representatives S of the double cosets AsB in F,

$$\sum_{s \in S} \overline{rank}(A \cap sBs^{-1}) \leq \overline{rank}(A) \cdot \overline{rank}(B).$$

We generalize it to virtually free groups (i.e., groups with finite index free subgroups).

Theorem (Klyachko-Zakharov, 2021)

Let G be a virtually free group G containing a free group F as a finite-index subgroup. Then for any subgroups A and B of G and for any system of representatives S of the double cosets AsB in G,

$$\sum_{s\in S} \overline{rk}(A\cap sBs^{-1}) \leq |G:F|\cdot \overline{rk}(A)\cdot \overline{rk}(B).$$

In particular, $\overline{rk}(A\cap B) \leq |G:F|\cdot \overline{rk}(A)\cdot \overline{rk}(B)$.

Here $\overline{rk}(H)$ is the *virtual reduced rank* of a virtually free group:

$$\overline{rk}(H) = \frac{1}{|H{:}K|}\cdot \overline{rank}(K) = \frac{1}{|H{:}K|}\cdot max(0, rank(K)-1),$$

where K is a finite-index free subgroup of H.

We further generalize it to virtually free products of left-orderable groups, thus generalizing also an earlier result by Antolin, Martino and Schwabrow of 2011 (which is itself a generalization of Mineyev-Friedman theorem for free products of left-orderable groups).
Here (some variation of) Kurosh rank plays the role of the free group rank.

In the proof we use some (reformulated) Mineyev's ideas, group actions on forests (instead of trees) and the following simple lemma about orbit intersections:

Lemma

Let A and B be subgroups of a group G acting freely on a set X, and let D be a set of distinct representatives of double cosets AgB. Then, for any A-invariant set $Y \subseteq X$ and any B-invariant set $Z \subseteq X$, the sum of the number of $(A^d \cap B)$-orbits in $(d^{-1}Y) \cap Z$ is not greater than the number of A-orbits in Y times the number of B-orbits in Z.